Logic in Medicine

An outline of formal logic and its applications in medicine—I

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I—Introduction

Once upon a time the educated man (or, more rarely, woman) would have in his or her intellectual background, along with many Greek verbs and other curiosities, a smattering of logic. Logic, advertised as revealing the "laws of thought," was mainly a theory of syllogistic inference dating back as a system to the fourth century BC and in particular to Aristotle. It produced such useful and decorative specimens as:

All Brazilians are footballers
All footballers are bipeds
Therefore all Brazilians are bipeds

—an instance of the valid syllogistic form known as "Barbara"[1] and enabled initiates to recognise and avoid the "undistributed middle" (a formal fallacy, not an unsightly condition brought on by hunching over textbooks). Its applicability to real life, however, was doubtful. Firstly, most of the really interesting reasoning going on was too advanced to be caught in the coarse mesh of a "tissue of syllogisms" being, for example, mathematical or analogical. Secondly, thought kept refusing to obey the "laws": only by jumping to conclusions, bending definitions, and the like can important progress be made in theorising, so that in describing anything like science logic labours along far behind life. Long before the intellectual explosion of the late nineteenth century that gave birth to modern medicine the study of formal logic had become an ingrown phenomenon. What kept it alive was not so much any theory that it offered of inference or rationality as the seductiveness of the patterns it made.

Divergence of medicine and logic

In the twentieth century logic and medicine have taken divergent paths. While medicine has emerged as a scientific discipline, sustained by unprecedented empirical success, merging at the edges not only with biology but with a range of sciences from chemistry to psychology, and serving as a focus for technological innovation, logic has become an extremely abstract subject mainly serving the needs of pure mathematics and as distant from practical motivations as any academic concern. In part, these different directions are historically explicable. Medicine responded to the possible when scientific theories of the origin and nature of diseases became well established and when modern drug technology and precision engineering emerged. Logic took the course it did because it was needed to help solve a deep crisis in the foundations of mathematics, and it stayed on to spawn new subjects of pure mathematical study.

This divergence was quite likely to have occurred anyway, given the nature of the subjects, under the pressure of the increasing depth of twentieth century inquiry with its consequent specialisation. It is wrong to bemoan the high degree of specialisation in current research. Although there is something disquieting about the thought of an extremely able mind engaged exclusively on a problem so minute that only the highly trained can see it, we should not forget that in a mature discipline only specialisation gets things done. What is genuinely pernicious is not narrowly delimited fields of inquiry but narrow vision blocking appreciation of anything outside the delimited field. I do not, of course, urge that medical practitioners all immediately take up mathematical logic or even suppose that a study of the mathematics of inference will make anyone a much better diagnostician—for that purpose empirical means tend to be more effective—but I do suggest that awareness of the abstract structure of theorising and decision making might contribute a degree of conceptual clarity, especially in those difficult circumstances in which the small steps of inference become important enough to be made explicit. I also want to indicate some ways in which recent developments in pure logic may be about to impinge on many other disciplines, medicine included.

Before this can be done it is necessary to describe formal logic. What follows is not intended to teach logic to anyone. There are many textbooks available,[1-3] and any interested reader should consult one of these for a proper introduction. Here I give only an overview, sketching in the conceptually important features in such a way that the later remarks make sense.

Valid or invalid?

Logic is concerned with arguments. An argument, like the one above about Brazilians and their feet, is not a dialogue but a record of a possible inference and consists of premises (two in the example, but any number, zero or more, in principle) and a conclusion, usually joined by "so" or "therefore." The premises and conclusion are statements and may be either true or false. An argument is valid if and only if there is no way that its premises could be true and its conclusion at the same time false. In a valid argument it is a matter of necessity that if the premises are true then so is the conclusion. Conversely, the argument is invalid if there is some possible situation that would make the premises true and the conclusions false. It is very important that there are valid arguments with false conclusions (and of course false premises):

Pope John Paul II is a Scot
All Scots support Rangers
Therefore Pope John Paul II supports Rangers.

There are also invalid arguments whose premises and conclusions are all true:

Pope John Paul II is a Christian
All Catholics are Christians
Therefore Pope John Paul II is a Catholic.

(Note that if the second premise were "All Christians are Catholics"

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This series is edited by Professor Calbert I Phillips.

the argument would be valid, though its premises would hardly be persuasive.) What there cannot be is a valid argument with true premises and a false conclusion.

Formal logic is the study not of individual arguments like those above but of abstractable features of language that enter systematically into questions of the validity or invalidity of argument forms. An argument form is the result of substituting variables for names, predicates, or even whole statements in an actual argument. It is valid if and only if every argument of that form is valid. Thus the above argument about Rangers is of the valid form:

n is S
All S are R
Therefore n is R.

A certain amount of chopping and squeezing is needed to get natural arguments to fit such schematic forms. To judge the acceptability or unacceptability of the paraphrasing required, there is no alternative to native speakers' intuitions—a fact that, once noted, robs formal logic of the inexorability popularly associated with it.

The method used in formal logic is first to move away from everyday reasoning, setting up completely abstract mathematical systems codifying inference in simple artificial languages, and then to seek importance for the results by mapping back on to natural languages like English. The intended reading of a logical system will, of course, influence the choice of its language and rules but is not intrinsic to the system any more than possible physical applications of such mathematical constructions as group theory or geometry are parts of those properly mathematical theories.

II—Ifs and ands: a formal calculus

To illustrate the notion of a formal language for logic, and to be able to make certain observations later, I shall now present a simple logical system suitable for analysing the parts played in argument by two constructions: the conjunction "and" and the conditional "if... then...." For these I shall use the notations "&" and "$\rightarrow$," respectively. In the calculus given here we are not interested in the internal structure of statements not constructed with & and $\rightarrow$, so we shall take an arbitrary set of such statements as atoms and write them as single letters, with superscripts if necessary: P, Q, R, P', Q', R', P''... The connectives & and $\rightarrow$ may be applied repeatedly to these atoms to build up formulas of any complexity—for example, $P \& Q$, $R \rightarrow R$, $(P \& R) \rightarrow (Q \rightarrow (P \& P))$, and so on. Here we have a language, a bit limited as languages usually go but sufficient to illustrate a few points.

We want to capture the valid forms of argument in our formal language. Because compounding under connectives is unlimited we should expect infinitely many valid forms, so listing them is not going to be helpful. What we do, therefore, is appeal to the notion of a formal derivation. To show that a given conclusion A follows from a set X of premises we produce a list $D_1, \ldots D_n$, A ending with A, each item in which is either one of the premises in the set X, or an immediate consequence derived from items earlier in the list by one of a small set of rules, or a subderivation in its own right. The rules defining "immediate consequence" and "sub derivation" will be specified below. First note how natural the idea of derivation is. An argument that may be complicated and unobvious is broken up by interpolating many small steps, each of which is simple and obvious and which only cumulatively provide the effect of complexity. Thus infinitely many argument forms may be reduced to a few very simple ones. This is the power of formal logic.

Ordinary proofs in mathematics are derivations in much the same sense given here. The argument whose premises are the axioms of Euclid's geometry and whose conclusion is Pythagoras's theorem, for example, is logically valid but far from obviously so; to make it convincing we interpose many simpler arguments whose validity is not in doubt. Ultimately, the derivation can be reconstructed in pure logic (though it needs a more elaborate system than the fragment presented here).

Rules of calculus

The precise rules governing & and $\rightarrow$ are fairly easily stated and justified. Firstly, any formula of the form $A \& B$ has the joint force of A and B. So:

Rule 1 A is an immediate consequence of $A \& B$.
Rule 2 B is an immediate consequence of $A \& B$.
Rule 3 $A \& B$ is an immediate consequence of A and B taken in either order.

These rules are given for all formulas A and B.

Secondly, to assert $A \rightarrow B$ (if A then B) is to claim a warrant for asserting B, given A. We can assert "If A then B" when we are in a position to infer B from A, perhaps together with other information we possess. So we may take it that the conditional $A \rightarrow B$ follows from the premises of an argument provided that if we took A as an extra premise we could derive B. This motivating thought gives rise to two more rules:

Rule 4 B is an immediate consequence of $A \rightarrow B$ and A in either order.
Rule 5 $A \rightarrow B$ is an immediate consequence of a subderivation with assumption A and conclusion B.

A subderivation is simply a derivation within a derivation, except that it must start with exactly one assumption (which may be any formula), whereas the main derivation starts with assumptions of all the premises. Items from any derivation may be used within any of its later subderivations, but items inside subderivations are not available from outside. Subderivations may be nested one inside the other to any finite depth.

Consider a couple of sample derivations to make all this clearer. First take the argument:

If ice is placed in water it floats
If ice is placed in water it melts
Therefore if ice is placed in water it floats and melts.

This is boring but valid, as it can be regimented to fit the form $P \rightarrow Q$, $P \rightarrow R$, therefore $P \rightarrow (Q \& R)$.
To prove this to be valid we derive its conclusion from its two premises thus:

(1)	$P \rightarrow Q$	Premise
(2)	$P \rightarrow R$	Premise
(3.1)	│ P	Assumption
(3.2)	│ Q	From 1 and 3.1 by rule 4
(3.3)	│ R	From 2, 3.1 by rule 4
(3.4)	│ $Q \& R$	From 3.2 and 3.3 by rule 3
(4)	$P \rightarrow (Q \& R)$	From 3, by rule 5

Item three of the main derivation here is a subderivation with assumption P and conclusion $Q \& R$. It is in turn composed of four items, 3.1 to 3.4, and is indented with a vertical line to make clear that it is a subderivation. Next consider the argument form $(P \& Q) \rightarrow R$, therefore $P \rightarrow (Q \rightarrow R)$.

(1)	$(P \& Q) \rightarrow R$	Premise
(2.1)	│ P	Assumption
(2.2.1)	│ │ Q	Assumption
(2.2.2)	│ │ $P \& Q$	From 2.1 and 2.2.1 by rule 3
(2.2.3)	│ │ R	From 1 and 2.2.2 by rule 4
(2.3)	│ $Q \rightarrow R$	From 2.2 by rule 5
(3)	$P \rightarrow (Q \rightarrow R)$	From 2 by rule 5.

Here item 2.2 (composed of items 2.2.1 to 2.2.3) is a subderivation of item 2, which in turn is a subderivation of the main proof. The converse argument form $P \rightarrow (Q \rightarrow R)$, therefore $(P \& Q) \rightarrow R$ is also valid. The two together show the equivalence, for example, of: "If more blood is lost and no transfusion given the patient will die" to "If more blood is lost the patient will die without a transfusion."

This is not an appropriate place to dwell on the details of the fragment of formal logic just presented. Any interested reader will find other thorough expositions in a similar style.[2,3] The reason for giving it here is to provide an example, a target for pointing, to sustain the discussion that follows. Before passing to that discussion we should note the important concepts of interpretation, model, and theory.

Interpreting theories

To interpret a formal system is to give it a reading by assigning some values to the formulas (and in more advanced cases to other things, such as names and predicates). This will then produce an account of the meaning of the logical symbols (& and → in the example) in terms of their effects on the values of formulas and so on. Every interpretation picks out some set of formulas as being "true." If every formula in a set S is true for a given interpretation then that interpretation is said to be a model of S. A theory, in the sense provided by a formal logic, is a set of formulas in the appropriate language such that whenever any argument from $A_1 \dots A_n$ to B is valid in the logic, and all of $A_1 \dots A_n$ are in the theory, so is B. The items in a theory are called its theorems. A model of a theory is therefore an interpretation for which everything said by that theory is true.

For example, we can interpret our calculus of & and → in terms of the concept of "information." Suppose that there are some "pieces of information" and that we can pick out various sets of these as possible "states of information." It does not matter formally what these are: only the structure of the idea counts. One possible state of information—call it T (for True)—is supposed to be the information actually given by the real world. Now each atom (P, Q, and so on) is interpreted as conveying a piece of information. For a given interpretation in this sense each formula is either "warranted" or "not warranted" by each state of information as follows:

(1) An atom is warranted by state S if and only if its information is in S.

(2) A conjunction A & B is warranted by S if and only if both of A and B are warranted by S.

(3) A conditional A→B is warranted by S if and only if B is warranted by every state that includes S and warrants A.

A formula is true for an interpretation if and only if it is warranted by T according to that interpretation. An argument form is valid provided that its conclusion is true for every interpretation for which its premises are true. It can be shown that validity thus defined coincides with derivability according to the five rules given above in the sense that we get the same set of valid argument forms whether we define the logic as a system of derivations or as a theory of information. This fact is a completeness theorem for the system in question. This system differs slightly from the one more usually found in introductory texts, as will be noted below in section IV.

What has been set out in this section is, of course, a very small part of formal logic. It can be elaborated to take account of much more complex reasonings, including arguments of the kind given in section I and many others. Logical theory since 1900 has been partly a matter of formulating such elaborations and partly concerned with the investigation of concepts arising—the theory of sets, model theory, proof theory, parts of abstract algebra, recursion theory, and so on. Next week we will leave aside the technicalities of mathematical logic and return to considering the ways in which logic has to do with medicine.

References

1 Kneale W, Kneale M. *The development of logic*. Oxford: Oxford University Press, 1962:232.
2 Fitch FB. *Symbolic logic; an introduction*. New York: Ronald Press, 1952.
3 Kalish D, Montague R. *Logic: techniques of formal reasoning*. New York: Harcourt, Brace and World, 1964.

Contemporary Themes

Waiting list statistics. II: an estimate of inflation of waiting list length

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Abstract

The discrepancy between the length of the waiting list and eventual admissions from the waiting list was investigated by comparing data from two different sources of routine statistics in the Oxford region. It was estimated that about 28% of the waiting list comprised patients who were not eventually admitted to hospital within the region.

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Introduction

Several studies have indicated that routine returns on the number of patients on waiting lists overestimate the numbers of patients who will eventually be admitted to hospital.[1-4] The Department of Health and Social Security has asked health authorities to review and validate their inpatient waiting lists and has suggested that at least one tenth of all patients on waiting lists will not eventually require admission.[5] We investigated the discrepancy between the length of the waiting list and the number of eventual admissions from the waiting list by comparing the data from two different sources of routine statistics.

Methods

We studied figures for general surgery, trauma and orthopaedic surgery, ear, nose, and throat surgery, gynaecology, ophthalmology, and plastic surgery from 1974 to 1983 in the Oxford region. These specialties accounted for about nine tenths of the patients on the waiting list. Data were included for the five districts in the region that collected data for the Hospital Activity Analysis, including reasonably complete data on waiting times, during the whole study.